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Interval competition graphs of symmetric digraphs*

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Abstract

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The competition graph of a loopless symmetric digraph H is the *two-step graph*, $S_2(H)$. Necessary and sufficient conditions on H are given for $S_2(H)$ to be interval or unit interval. These are useful properties when application requires that the competition graph be efficiently colorable. Computational aspects are discussed, as are related open problems.

In this paper we answer the following question: given a loopless symmetric digraph D with underlying interval graph H, what conditions are necessary and sufficient for the competition graph of D to be interval or unit interval? This question, first posed by Raychaudhuri and Roberts [10], was left open in our previous paper [7]. In that paper we presented the background of this question in the context of the channel assignment problem as well as recent results on several questions posed by Raychaudhuri and Roberts [9, 10]. Here we extend those results. We begin by stating definitions, and results from our previous work, that will be useful here. Proofs will be omitted; the interested reader may consult [7].

Suppose that $D=(V,A)$ is a digraph and B and C are (not necessarily disjoint) sets of vertices in D. The *generalized competition graph* $G(D,B,C)$ corresponding to D, B,

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and C is an undirected graph with vertex set B and an edge between two distinct vertices $x, y \in B$ if and only if for some $z \in C$, (x,z) and (y,z) belong to A. If $B = C = V$ then we call G the *competition graph*, denoted $G(D)$. For a survey of recent results on competition graphs, see Lundgren [5].

A graph H is an *interval graph* if it is the intersection graph of a family of intervals on the real line. If these intervals are closed and of unit length, then we say H is *unit interval*. Useful characterizations of interval graphs were given by Gilmore and Hoffman [3] and by Lekkerkerker and Boland [4], and are used repeatedly in the proofs that follow.

For a graph H, we denote its *two-step graph* by $S_2(H)$. In $S_2(H)$, $[x, y]$ is an edge if and only if there is a path of length two between x and y in H. The *open neighborhood* $N(v)$ of a vertex v in a graph H is the set of all vertices adjacent to v. The *closed neighborhood* $N[v]$ of v is the set $N(v) \cup \{v\}$. A *simplicial* vertex v in a graph H has the property that $N(v)$ is a clique.

Maehara [8] has shown that every graph may be viewed as the intersection graph of closed unit spheres in Euclidean k-space for k sufficiently large. The *sphericity* of a graph is the smallest such k. Roberts [12] defines the *boxicity* of a graph in similar fashion, where a *box* in Euclidean k-space is a generalized rectangle with sides parallel to the coordinate axes. An incomplete interval graph H has boxicity 1; if H is unit interval then it has sphericity 1. Thus, our main question may be viewed as a special case of the more general question: What are the generalized competition graphs of symmetric digraphs whose underlying graph H has boxicity or sphericity at most k?

Theorem 1. *Suppose that D is loopless symmetric digraph and H is the undirected graph corresponding to D. Then the competition graph G of D is two-step graph $S_2(H)$.*

A family of sets $\mathcal{S} = \{S_1, \ldots, S_r\}$ of vertices of H will be called a *competition cover* of H if the following conditions are satisfied:

(i) If $i, j \in S_m$ for some m then $i, j \in N(k)$.

(ii) If $i, j \in N(k)$ for some k, then $i, j \in S_m$ for some m.

A competition cover of a digraph D is similarly defined, with $i, j \in N(k)$ replaced by $(i, k), (j, k) \in A(D)$ in (i) and (ii). Observe that if $\mathcal{S}$ is the set of open neighborhoods of H, then $\mathcal{S}$ is a competition cover, but other sets will also work. Also note that $\mathcal{S}$ is not necessarily a cover in the usual sense, i.e., a vertex or edge cover. $\mathcal{S}$ covers the *competitions* in H. We say that a ranking $S_1, \ldots, S_r$ of a competition cover of H is *consecutive* if whenever a vertex x is in S_i and S_j for $i < j$, then for all $i < k < j$, $x \in S_k$.

Theorem 2. *$S_2(H)$ is an interval graph if and only if H has a competition cover $\mathcal{S}$ that has a consecutive ranking.*

Theorem 2 follows directly from a more general result of Lundgren and Maybee [6]. In our previous work, we were able to use Theorem 2 to show that certain classes of interval graphs have interval two-step graphs. Application of the theorem was limited by the need to construct a competition cover for each class of interval graphs. We now develop a competition cover that works for all classes of loopless interval graphs. In the sequel we shall assume that all graphs are connected and loopless, since disconnected graphs and graphs with loops are uninteresting in the context of competition. For the same reason, we do not consider complete graphs. We first classify the nonsimplicial vertices of a graph G and then define a special competition cover of G.

(i) Let H be a connected incomplete graph with maximal cliques $C_1, \dots, C_m$ and let $v \in V(H)$ be a nonsimplicial vertex. If v is contained only in maximal cliques C_i where $|C_i| \geqslant 3$ then call v a type I vertex. If v is contained only in maximal cliques of size $|C_i| = 2$ then call v a type II vertex. If v is contained in maximal cliques of both size classes then call v a type III vertex.

(ii) Let H be as above, with nonsimplicial vertices $v_1, \dots, v_k$. For each nonsimplicial v_i, construct the set(s) S_i as follows: If v_i is type I then $S_i = N[v_i]$. If v_i is type II then $S_i = N(v_i)$. If v_i is type III then let $S_{i_1} = \bigcup \{C_j \mid v_i \in C_j, |C_j| \geqslant 3\}$ and let $S_{i_2} = N(v_i)$.

For example, consider the graph H of Fig. 1. The nonsimplicial vertices of H are b (type II), c (type III), and e (type I). The sets constructed are then $S_b = \{a, c\}$, $S_{c_1} = \{c, d, e\}$, $S_{c_2} = \{b, d, e\}$, and $S_e = \{c, d, e, f, g\}$.

Lemma 1. *Let H be a connected incomplete graph. Let $S(H)$ be the collection of sets defined above. Then $S(H)$ is a competition of H.*

Proof. We must show that $p, q \in S_k$ for some k if and only if $p, q \in N(r)$ for some r.

First suppose that $p, q \in S_k$ for some k. Let v_k be the nonsimplicial vertex associated with S_k in the construction of $S(H)$. If either $p = v_k$ or $q = v_k$, then v_k is either type I or type III; if v_k is type III, then $S_k = S_{k_1}$. In either case, there exists some vertex r such that $p, q \in N(r)$. If $p \neq v_k$ and $q \neq v_k$, then $p, q \in N(v_k)$. Thus, if $p, q \in S_k$ for some k then $p, q \in N(r)$ for some r.

Now suppose that $p, q \in N(r)$ for some r. If r is a nonsimplicial vertex, then $p, q \in S_r, S_{r_1}$, or S_{r_2}, depending on the type of r. If r is a simplicial vertex, then r is adjacent to some nonsimplicial vertex t, where t is either type I or type III, so $p, q \in S_t$ or $p, q \in S_{t_1}$. Thus, $p, q \in N(r)$ for some r then p, q in S_k for some k.

Therefore, $S(H)$ is a competition cover. □

We shall call $S(H)$ a *nonsimplicial competition cover*. The proof of the next lemma follows immediately from the above.

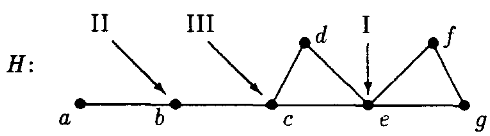

Fig. 1. Classification of nonsimplicial vertices.

Lemma 2. *Let $S'(H)=\{S_i \in S(H) \mid S_i$ is maximal$\}$. Then $S'(H)$ is a competition cover of H.*

We shall call $S'(H)$ the *maximal nonsimplicial competition cover* of H.

Lemma 3. *Let H be a connected incomplete interval graph. Let $S(H)$ be the nonsimplicial competition cover of H, $S_i \in S(H)$, and v_i, the vertex associated with S_i. If x is any vertex such that $[x, v_i]$ is not an edge, then the set $C=\{v \in S_i \mid [s, v] \in E(H)\}$ is a clique.*

Proof. Let $p, q \in S_i$, such that $[x, p]$, $[x, p] \in E(H)$. H is interval, hence chordal. Since $[x, v_i]$ is missing $[p, q] \in E(H)$. $\square$

Lemma 4. *Let H be a connected incomplete interval graph. Let $S'(H)=\{S_1, \ldots, S_m\}$ be the maximal nonsimplicial competition cover of H. Let $x \in V(H)$. If x is connected by a path of length two to every vertex in some $S_i \in S'(H)$, then $x \in S_i$.*

Proof. Let v_i be the vertex associated with S_i.

Case 1: v_i is type I. Then S_i contains at least two maximal cliques, each containing v_i.

(i) First suppose that x is adjacent to no vertex in S_i. Then x cannot be connected to v_i by a path of length two, a contradiction. Therefore, x must be adjacent to at least one vertex in S_i.

(ii) Now suppose x is adjacent to one or more vertices of exactly one maximal clique in S_i, say C. Let $p \in C$ and $[x, p] \in E(H)$. Clearly, x is not adjacent to v_i, since then x would be adjacent to every maximal clique of S_i, and so $p \neq v_i$. Since x is also connected to p by a path of length two, there exists q such that $[x, q]$ and $[p, q]$ are edges. If $q \in C$, we have the graph H_1 of Fig. 2.

Note that there exists some vertex $r \in S_i \setminus C$, since v_i is type I, and that r cannot be adjacent to any vertex adjacent to x in C. If, for example, $[p, r] \in E(H)$ then p would be contained in at least two maximal cliques containing v_i. So $[p, r]$ and $[q, r]$ are not edges, but then there exists some vertex $t \notin S_i$ such that $[x, t]$ and $[r, t]$ are edges. Clearly, $[t, v_i] \notin E(H)$ since $t \notin S_i$. If $[p, t]$ ($[q, t]$) is an edge then $\langle p, t, r, v_i \rangle$ ($\langle q, t, r, v_i \rangle$) is an induced 4-cycle; if $[p, t]$ ($[q, t]$) $\notin E(H)$ then we have an induced 5-cycle

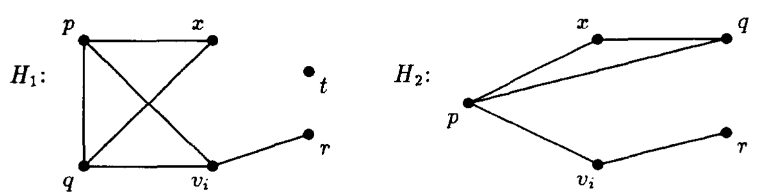

Fig. 2. Graphs for case 1(ii) of Lemma 4.

$\langle p, x, t, r, v_i\rangle$ ($\langle q, x, t, r, v_i\rangle$). In either case, we contradict the assumption that H is interval.

Now if $q \notin C$, then $q \notin S_i$, and we have the graph H_2 of Fig. 2. As before, $[p, r] \notin E(H)$. Also $[q, v_i] \notin E(H)$. Now x must be connected to r by a path of length two. If $[q, r] \in E(H)$ then $\langle q, r, v_i, p\rangle$ is an induced 4-cycle; if $[q, r] \notin E(H)$ then there exists some t such that either $\langle x, t, r, v_i, p\rangle$ or $\langle t, r, v_i, p\rangle$ is an induced cycle of length at least 4, contradicting the assumption that H is interval.

Therefore, x cannot be adjacent to vertices of exactly one maximal clique is S_i.

(iii) Suppose that x is adjacent to vertices of exactly one maximal clique is S_i. If $[x, v_i] \in E(H)$ then we are done: $x \in S_i$. So assume $[x, v_i] \notin E(H)$. Let $C_x = \{v \in S_i \mid [x, v] \in E(H)\}$. By Lemma 3, C_x is a clique, so $C_x \subset C_k$ for some maximal C_k containing v_i. Since v_i is type I, C_k is not the only maximal clique in S_i.

Claim. If C_j is any other maximal clique in S_i then $C_j \cap C_x \neq \emptyset$. Suppose not. It is clear that every vertex $v \in C_j$ must be adjacent to some vertex in C_x, for otherwise we would have edges $[x, t]$ and $[t, v]$, where $t \notin S_i$. We then obtain an induced cycle $\langle x, t, v, v_i, p\rangle$ or $\langle t, v, v_i, p\rangle$, but either cycle contradicts the assumption that H is interval. This is shown in the graph H_1 of Fig. 3. Now choose $r \in C_j$ such that r is adjacent to as few vertices of C_x as possible. By our choice of $r, r \notin C_k$. Choose $p \in C_x$ such that $[r, p] \in E(H)$, $s \in C_j$ not adjacent to p, and $q \in C_x$ not adjacent to r. Note that both s and q exist, since $p \notin C_j$ and $r \notin C_k$. If $[s, q] \in E(H)$ then we obtain an induced 4-cycle $\langle s, q, p, r\rangle$, as in graph H_3 of Fig. 3, so assume that $[s, q] \notin E(H)$. We have already observed that s must have at least one neighbor $t \in C_x$. If $[t, r] \notin E(H)$, then we obtain an induced 4-cycle $\langle s, t, p, r\rangle$, as in graph H_3 of Fig. 3. Thus, for every $t \in C_j$

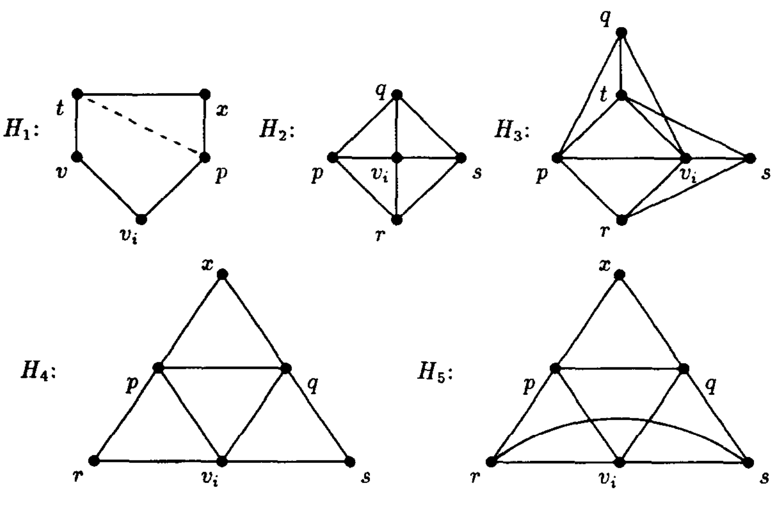

Fig. 3. Subgraphs for case I(iii) of Lemma 4.

satisfying $[t,s]\in E(H)$ we have $[t,r]\in E(H)$. But r is also adjacent to p, so r is adjacent to more vertices in C_x than s, a contradiction.

We conclude that $C_j\cap C_x\neq\emptyset$, proving the claim.

Now if some vertex adjacent to x, say p, is contained in every maximal clique of S_i, then $S_p\supset S_i\cup\{x\}$, contradicting maximality of S_i. This forces S_i to contain at least two maximal cliques in addition to C_k, say C_j and C_m. Each contains at least one vertex of C_x, and these vertices are distinct. Let $p\in C_j\cap C_x$, $p\notin C_m$, $q\in C_x\cap C_m$, $q\notin C_j$. Letting $r\in C_j\backslash\{C_k\cup C_m\}$ and $s\in C_m\backslash\{C_k\cup C_j\}$, we may construct one of the induced subgraphs H_4, H_5 of Fig. 3. Neither is interval, contradicting the assumption that H is interval. Thus, $[x,v_i]\in E(H)$ and so $x\in S_i$. We conclude that if v_i is type I and x is connected by a path of length two to very vertex in S_i, then $x\in S_i$.

Case 2: v_i is type II.

If $[x,v_i]\in E(H)$, then $x\in S_i$, so assume not. Now v_i is adjacent to at least two vertices, say p and q, where $[p,q]\notin E(H)$ since v_i is type II. Since x is connected to p and q by paths of length two, then we have paths $[x,r,p]$ and $[x,s,q]$, where possibly $r=s$. Note that $[r,v_i]$ and $[s,v_i]$ are forbidden, since v_i is type II. If $r=s$ then we have an induced 4-cycle $\langle r,p,v_i,q\rangle$; if $r\neq s$, then $\langle x,r,p,v_i,q,s\rangle$ contains at least one induced cycle of length at least four, but this is a contradiction since H is interval. Thus, $[x,v_i]\in E(H)$, and so $x\in S_i$. We conclude that if v_i is type II and x is connected to every vertex of S_i by a path of length two, then $x\in S_i$.

Case 3: v_i is type III.

(i) Suppose $v_i\notin S_i$. Then $S_i=S_{i_2}$. If $[x,v_i]\in E(H)$ then $x\in S_i$ and we are done, so assume not. Since v_i is type III we may find at least one maximal clique C_k, $|C_k|\geqslant 3$, and at least one maximal clique C_j, $|C_j|=2$, both containing v_i. Let $p\in C_k$ and $r\in C_j$; note $[p,r]\notin E(H)$. There exists a path of length two from x to r not using v_i, so there exists some vertex q such $[x,q]$, $[q,r]\in E(H)$. Note that $q\notin S_i$, by our choice of r. Now either q is connected by an edge to $C_k\backslash\{v_i\}$, or x is connected to C_k by another path; either case we have an induced cycle of length at least four, a contradiction. Therefore, $[x,v_i]\in E(H)$ and so $x\in S_i$.

(ii) Now suppose $v_i\in S_i$. Then $S_i=S_{i_1}$. If $[x,v_i]\in E(H)$ then we done; $x\in S_i$. So assume $[x,v_i]\notin E(H)$. Let C_j, C_k, p, and r be as described above. Since $|C_k|\geqslant 3$, we may choose an additional vertex, say $q\in C_k$. If $[x,r]\in E(H)$, then we get an induced k-cycle, $k\geqslant 4$, in connecting x to C_k. So, x is connected by an edge to *no* vertex that is contained in exactly one maximal clique of size two, and we may therefore ignore such cliques.

In view of this, if S_i contains at least two maximal cliques of size at least three, we may apply the type I proof verbatim. So assume that S_i contains exactly one maximal clique of size at least three, namely C_k. Either x is adjacent to at least two vertices in C_k, say p and q, or x is adjacent to exactly one vertex in C_k, say p, and is connected to p by a path of length two using some $s\notin S_i$. In either case, either $S_p\supset S_i\cup\{x\}$ or $S_{p_1}\supset S_i\cup\{x\}$, contradicting maximality of S_i. Thus $[x,v_i]\in E(H)$, and so $x\in S_i$. We conclude that if v_i is type III and x is connected to every vertex of S_i by paths of length two, then $x\in S_i$. This completes the proof of Lemma 4. $\square$

Theorem 3. *Let H be connected, incomplete interval graph and let $S'(H)$ be the maximal nonsimplicial competition cover of H. Then $S_2(H)$, the competition graph of H, is interval if and only if $S'(H)$ has a consecutive ordering.*

Proof. First suppose that $S_2(H)$ is interval. Then we may find a consecutive ordering of the maximal cliques of $S_2(H)$. By Lemma 4, every set $S_i \in S'(H)$ induced a maximal clique in $S_2(H)$. Depending on the type of v_i, the vertex associated with S_i, both S_i and the maximal clique that it induces in $S_2(H)$ may or may not contain v_i. In either case, if the maximal which corresponds to $S'(H)$. Thus $S'(H)$ has a consecutive ordering.

Now suppose that $S'(H)$ has a consecutive ordering. By Lemma 2 and Theorem 2, $S_2(H)$ is an interval graph. □

Roberts [11] showed that an interval graph is a unit interval graph if and only if it does not contain $K_{1,3}$ as an induced subgraph. It is useful, then to determine when $S_2(H)$ contains an induced $K_{1,3}$.

Theorem 4. *Let G be a graph. Then $S_2(G)$ has an induced $K_{1,3}$ if and only if G contains an induced subgraph that is itself either a subgraph of H_1 or H_2, containing an embedded $H_{1,0}$, or a subgraph of H_3, containing an embedded $H_{2,0}$, where these graphs are shown in Fig. 4 and the embedding of $H_{1,0}$ in H_1 and H_2 is as indicated by the heavier edges.*

Proof. Refer to Fig. 5. Suppose H_1 is labeled as shown. Then $S_2(H_1)$ is as indicated, and contains a $K_{1,3}$ induced by $\{x, y_1, y_2, y_3\}$. Moreover, any induced subgraph H' of H_1 containing $H_{1,0}$ will also induce a $K_{1,3}$ in $S_2(H')$.

Now consider the indicated labeling of H_2 and the corresponding $S_2(H_2)$. Note that $\{x, y_1, y_2, y_3\}$ induces a $K_{1,3}$ in $S_2(H_2)$; any subgraph H' of H_2 containing $H_{1,0}$ must therefore induce a $K_{1,3}$ in $S_2(H')$.

Finally, consider the indicated labeling of H_3 and the corresponding $S_2(H_3)$. Again, $\{x, y_1, y_2, y_3\}$ induces a $K_{1,3}$, so any subgraph H' of H_3 containing $H_{2,0}$ induces $K_{1,3}$ in $S_2(H')$.

We must now show that any G inducing a $K_{1,3}$ in $S_2(G)$ must have one of the listed subgraphs. Let G be such a graph, and let the induced $K_{1,3}$ in $S_2(G)$ be labeled as

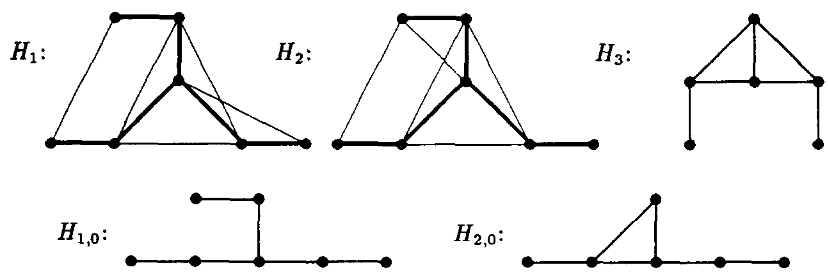

Fig. 4. Subgraphs for Theorem 4.

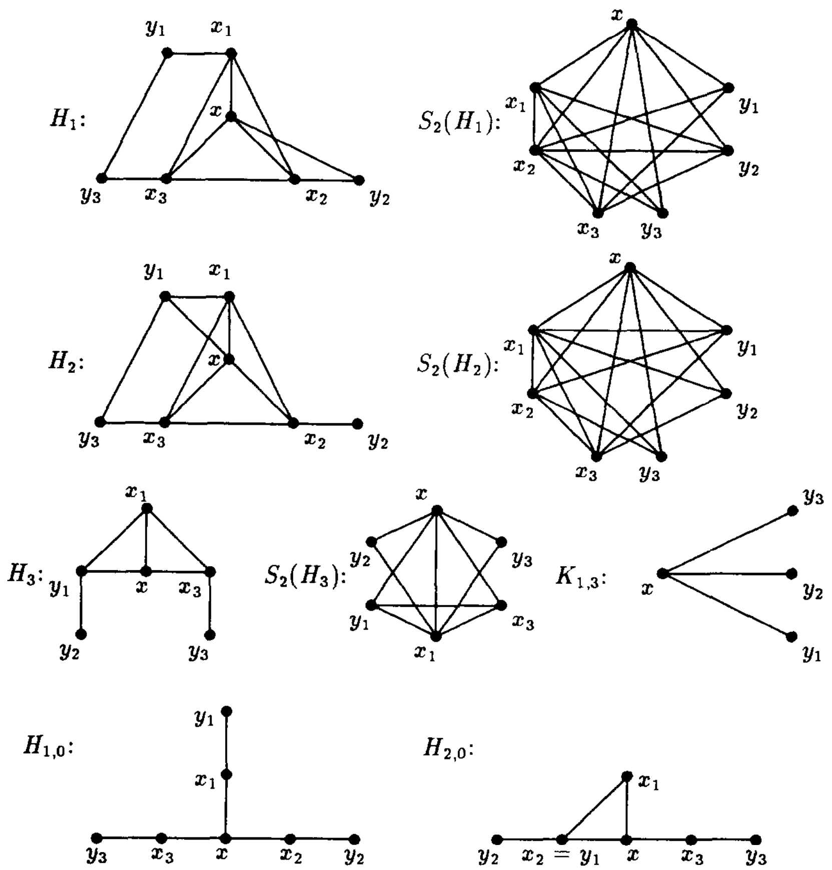

Fig. 5. Graphs for proof of Theorem 4.

shown in Fig. 5. Since we have edges $[x, y_i]$ on $S_2(g)$ for $i = 1, 2, 3$, then there must exist vertices x_i, $i = 1, 2, 3$, where $[x, y_i]$ is the edge in $S_2(G)$ induced by the path $[x, x_i, y_i]$ in G. Note that the x_i must be distinct, since otherwise some edge $[y_i, y_i]$ would be present in $S_2(G)$. We now have two possibilities.

Case 1: $K_{1,3}$ is an induced subgraph of $S_2(G)$ and $\{x_1, x_2, x_3\} \cap \{y_1, y_2, y_3\} = \emptyset$. We may arrange these in such a way that $[x, y_i]$ is the edge induced by $[x, x_i, y_i]$ for $i = 1, 2, 3$. Then G must contain $H_{1,0}$ as a subgraph, where this graph is labeled as shown in Fig. 5. Now if $[y_i, x_j]$ were an edge for any $i \neq j$ then the edge $[y_i, y_j]$ would be present in $S_2(G)$. If the edges $[x, y_j]$ and $[x, y_j]$ were both present in G, with $i \neq j$, then $[y_i, y_j]$ would be an edge in $S_2(G)$, so at most one edge $[x, y_i]$ can exist. Similarly, at most one edge $[y_i, y_j]$ can be present. Edges $[x_i, x_j]$ do not induce edges $[y_i, y_j]$; all are allowed. Thus, we obtain maximal preimages H_1, H_2.

Case 2: $K_{1,3}$ is an induced subgraph of $S_2(G)$ and $\{x_1, x_2, x_3\} \cap \{y_1, y_2, y_3\} = \emptyset$. Suppose that this intersection contained more than one element, say $x_1 = y_1$ and

$x_2 = y_2$. Then x is adjacent to both y_1 and y_2 in G, hence $[y_1, y_2] \in E(S_2(G))$. So $|\{x_1, x_2, x_3\} \cap \{y_1, y_2, y_3\}| = 1$. Without loss of generality, suppose that $x_2 = y_1$. Then $H_{2,0}$, as labeled in Fig. 5, must be present in G as a subgraph. This subgraph cannot contain either an edge $[y_i, x]$ for $i \neq 2$ or any edge $[y_i, y_j]$. Thus the only additional edge allowed is $[x_1, x_3]$ and we obtain the maximal preimage H_3. $\square$

We easily obtain the following corollary to Theorem 4, giving necessary and sufficient conditions for the two-step graph of a graph G to be unit interval.

Corollary 1. *Let H be a graph. Then $S_2(H)$ is unit interval if and only if H does not contain, as an induced subgraph, any of the subgraphs described in Theorem* 4.

Proof. $S_2(H)$ is unit interval if and only if it contains no induced $K_{1,3}$, by Roberts [11]. By Theorem 4, $S_2(H)$ contains no induced $K_{1,3}$ if and only if H does not contain, as an induced subgraph, any of the listed graphs. $\square$

Theorem 3 and Corollary 1 are potentially useful results in the context of the channel assignment problem, where one seeks optimal colorings or T-coloring of the conflict graph for transmitters/receivers. Cozzens and Roberts [2] show that there is an $O(n^2)$ algorithm for finding a T-coloring of a unit interval graph. Raychaudhuri [9] extended this result to interval graphs. The conditions of Theorem 3 can be tested in time proportional to $|V|^2$. Recognition that a graph H is interval, and enumeration of its maximal cliques, are linear in $|V| + |E|$ using the PQ-tree algorithm of Booth and Lueker [1]. From the clique matrix we may construct $S(H)$ in time proportional to $|V|^2$, and then apply the same algorithm to determine whether $S(H)$ has a consecutive ordering. Implementation of Corollary 1 appears less promising, because of the complexity of testing the forbidden subgraph conditions.

This work raises interesting questions whose solution is likely to have practical application to the channel assignment problem. Especially, useful would be the case where H has boxicity or sphericity at most 2 or 3 (an interval graph has sphericity 1), but we might consider other restrictions on H as well.

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